Equations defining the affine Grassmannian of SL_n

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A solved problem: Standard monomials on the finite Grassmannian

Let $V = \Bbbk^n$. The *Plücker embedding* realises the finite Grassmannian as a projective variety:

$$\mathrm{Gr}(r, n) = \{W \subseteq V \mid \dim W = r\} \xhookrightarrow{p} \mathbb{P}(\wedge^r V), \quad \mathrm{span}_\Bbbk\{v_1, \dots, v_r\} \mapsto [v_1 \wedge \dots \wedge v_r].$$

Coordinates on $\mathbb{P}(\wedge^r V)$ are labelled by the set $C_{r,n} = \{I \subseteq \{1, \dots, n\} \mid |I| = r\}$ of r-element subsets:

$$\Bbbk[\mathbb{P}(\wedge^r V)] = \Bbbk[x_I \mid I \in C_{r,n}], \quad \text{where } I = \{i_1 < \dots < i_r\} \text{ and } x_I \text{ is dual to } e_{i_1} \wedge \dots \wedge e_{i_r}.$$

The purpose of *standard monomial theory* is to describe a $\Bbbk$-basis of the homogeneous coordinate ring $\Bbbk[\mathrm{Gr}(r,n)] = \mathbb{K}[x_I \mid I \in C_{r,n}]/\mathcal{P}$, where $\mathcal{P} = \ker p^*$ is the Plücker ideal.

The monomial $x_I x_J x_K \in \Bbbk[x_I \mid I \in C_{r,n}]$ is a *standard monomial* if $I \leq J \leq K$ entrywise, (as a tableau, this means weakly increasing down the columns). Of course there are non-standard monomials, say if $I = \{1, 3, 6, 7\}$ and $J = \{2, 3, 4, 8\}$:

I and J are incomparable under $\leq$ (the problem is highlighted pink in the diagram) and so cannot be part of a standard monomial $x_I x_J x_K$. We will *straighten* $x_I x_J$ by finding a quadratic relation $P_{I,J} \in \mathcal{P}$ that contains $x_I x_J$ and vanishes on the embedded Grassmannian $\mathrm{Gr}(r = 4, n)$.

Split (I, J) into $A = (1, 3)$, $B = (2, 3, 4, 6, 7)$ and $C = (8)$ as above, and send $x_A \otimes x_B \otimes x_C$ through the map

$$\wedge^2 V \otimes \wedge^5 V \otimes \wedge^1 V \xrightarrow{1 \otimes \mathrm{comult}_{2,3} \otimes 1} \wedge^2 V \otimes \wedge^2 V \otimes \wedge^3 V \otimes \wedge^1 V \xrightarrow{\mathrm{mult}_{2,2} \otimes \mathrm{mult}_{3,1}} \wedge^4 V \otimes \wedge^4 V \twoheadrightarrow \mathrm{Sym}^2(\wedge^4 V)$$

to get a quadratic relation $P_{I,J}$ which includes $x_I x_J$. (comult is the *signed unshuffling* of the sequence):

$$x_{13} \otimes x_{23467} \otimes x_8 \mapsto x_{13} \otimes (x_{23} \otimes x_{467} - x_{24} \otimes x_{367} + x_{26} \otimes x_{347} - \dots + x_{67} \otimes x_{234}) \otimes x_8$$
$$\mapsto 0 + x_{1234}x_{3678} - x_{1236}x_{3478} - \dots + \underbrace{x_{1367}x_{2348}}_{x_I x_J} = P_{I,J}$$

$P_{I,J}$ vanishes on $\mathrm{Gr}(r, n)$ because of the $\wedge^{r+1}$ term coming from x_A, hence $P_{I,J} \in \mathcal{P}$. A more detailed inductive argument shows that any monomial $x_{I_1} x_{I_2} \cdots x_{I_\ell}$ can be straightened to a linear combination of standard monomials, hence the *standard monomials span* the ring $\Bbbk[x_I \mid I \in C_{r,n}]/\mathcal{P}$. A more careful argument shows they are linearly independent.

Our problem: Standard monomials on the affine Grassmannian $\mathrm{Gr}_{\mathrm{SL}_n}$

The affine Grassmannaian $\mathrm{Gr}_{\mathrm{SL}_n}$ admits an embedding i_n into the infinite Grassmannian $\mathrm{Gr}(\infty)$, which in turn embeds via the Plücker embedding p into the projectivisation $\mathbb{P}(\mathcal{F})$ of Fock space. Drawing analogies from above, $\mathrm{Gr}(\infty)$ is like $\mathrm{Gr}(r, n)$ and $\mathcal{F}$ is like $\wedge^r V$, however $\mathrm{Gr}_{\mathrm{SL}_n}$ is quite a different object.

$$\mathrm{Gr}_{\mathrm{SL}_n} \xhookrightarrow{i_n} \mathrm{Gr}(\infty) \xhookrightarrow{p} \mathbb{P}(\mathcal{F})$$

The ideal $\mathcal{P}$ cutting out $\mathrm{Gr}(\infty)$ inside $\mathbb{P}(\mathcal{F})$ is an infinite analogue of the Plücker relations. By a conjecture of Kreiman, Lakshmibai, Magyar, and Weyman [KLMW07] recently proven by Muthiah, Weekes, and Yacobi [MWY18], the set $\mathcal{S}_n$ of linear functions on $\mathcal{F}$ vanishing on $\mathrm{Gr}_{\mathrm{SL}_n}$ are given by the *shuffle equations.*

Problem: Confirm that $\mathcal{S}_n$ is the defining ideal of $\mathrm{Gr}_{\mathrm{SL}_n}$ inside $\mathrm{Gr}(\infty)$.

Approach: Develop a standard monomial theory for $\Bbbk[\mathrm{Gr}(\infty)]/\mathcal{S}_n$, and compare with a known basis for $\Bbbk[\mathrm{Gr}_{\mathrm{SL}_n}]$ given by FLOTW multpartitions.

Maya diagrams, semi-infinite wedges, and charged partitions

A *Maya diagram* $\mathsf{m} \colon \mathbb{Z} \to \{\circ, \bullet\}$ is a 2-colouring that is eventually white to the left and black to the right.

⋯	−6	−5	−4	−3	−2	−1	0	1	2	3	4	5	6	7	⋯
	○	○	○	●	○	○	●	●	○	●	○	●	●	●	

It can be recorded by the location of its white beads $\mathsf{m}^\circ \colon \mathbb{Z}_{<0} \to \mathbb{Z}$, or its black beads $\mathsf{m}^\bullet \colon \mathbb{Z}_{\geq 0} \to \mathbb{Z}$.

$$\mathsf{m}^\circ = (\dots, -6, -5, -4, -2, -1, 2, 4) \mid (-3, 0, 1, 3, 5, 6, 7, \dots) = \mathsf{m}^\bullet$$

The union $\mathsf{m}^\circledcirc \colon \mathbb{Z} \to \mathbb{Z}$ is a bijection, where $\mathsf{m}^\circledcirc(i) - i$ stabilises to the *charge* $c(\mathsf{m})$ (here $c(\mathsf{m}) = 1$).

i	⋯	−6	−5	−4	−3	−2	−1	0	1	2	3	4	5	6	⋯
$\mathsf{m}^\circledcirc(i)$	⋯	−5	−4	−2	−1	2	4	−3	0	1	3	5	6	7	⋯
$\mathsf{m}^\circledcirc(i) - i$	⋯	1	1	2	2	4	5	−3	−1	−1	0	1	1	1	⋯
$\mathsf{m}^\circledcirc(i) - i - c(\mathsf{m})$	⋯	0	0	1	1	3	4	−4	−2	−2	−1	0	0	0	⋯

The sequence $-(\mathsf{m}^\circledcirc(i) - i - c(\mathsf{m}))$ defines a partition $(4, 2, 2, 1, 0, 0, 0, \dots)$. The following are in bijection:

1. The *Maya diagram* $\mathsf{m} \colon \mathbb{Z} \to \{\circ, \bullet\}$ shown above, 2-colouring the integers.
2. The *semi-infinite wedge* $e_{-3} \wedge e_0 \wedge e_1 \wedge e_3 \wedge e_5 \wedge e_6 \wedge \cdots$ giving the sequence $\mathsf{m}^\bullet$.
3. The *charged partition* $(c, \lambda) = (1, (4, 2, 2, 1))$.

These three combinatorial objects all label the same basis of Fock space $\mathcal{F}$.

Fermionic Fock space

The *Fermionic Fock space* $\mathcal{F}$ is the vector space with basis given by Maya diagrams (or semi-infinite wedges, or charged partitions). It is graded by charge:

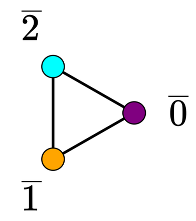
$$\mathcal{F} = \bigoplus_{c \in \mathbb{Z}} \mathcal{F}^{(c)}, \quad \text{where } \mathcal{F}^{(c)} = \mathrm{span}_\Bbbk\{(c, \lambda) \mid \lambda \in \mathsf{Partitions}\}.$$

The homogeneous coordinate ring is a polynomial ring in infinite variables: $\Bbbk[\mathbb{P}(\mathcal{F})] = \Bbbk[x_\mathsf{m} \mid \mathsf{m} \in \mathsf{Mayas}]$. Similarly to the finite case, we say that $x_{\mathsf{m}_1} \cdots x_{\mathsf{m}_\ell}$ is a *standard monomial* if $\mathsf{m}_1 \leq \dots \leq \mathsf{m}_\ell$, where the ordering $\leq$ is by containment of charged partitions.

The standard monomials form a $\Bbbk$-basis of $\Bbbk[\mathrm{Gr}(\infty)]$, however they do not appear to play nicely when the shuffle relations $\mathcal{S}_n$ are also introduced.

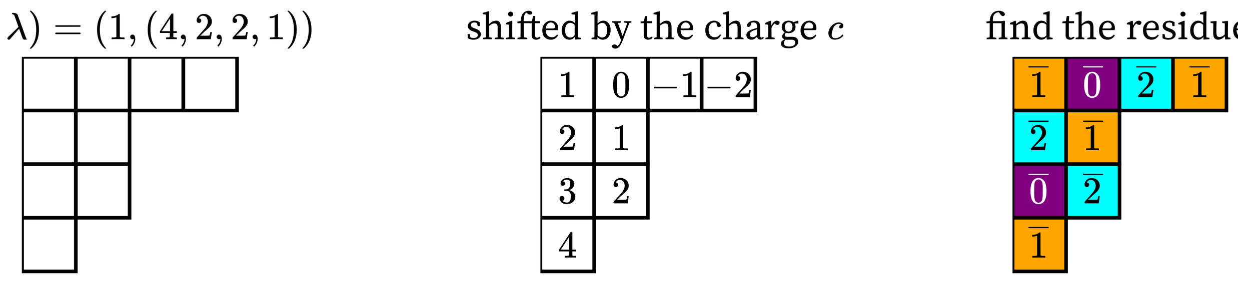
The action of $\widehat{\mathfrak{sl}_n}$ on Fock space, the representation $V(\Lambda_0)$

The Lie algebra $\widehat{\mathfrak{sl}_n}$ is the Kac-Moody algebra associated to a cycle diagram on n nodes. For example, $\widehat{\mathfrak{sl}_3}$ is generated by the *Chevalley generators* $E_\bullet, E_\bullet, E_\bullet, F_\bullet, F_\bullet, F_\bullet$, and the *derivation* $d \in \mathfrak{h}$ satisfying $[d, E_i] = \delta_{i,\bullet} E_i$.

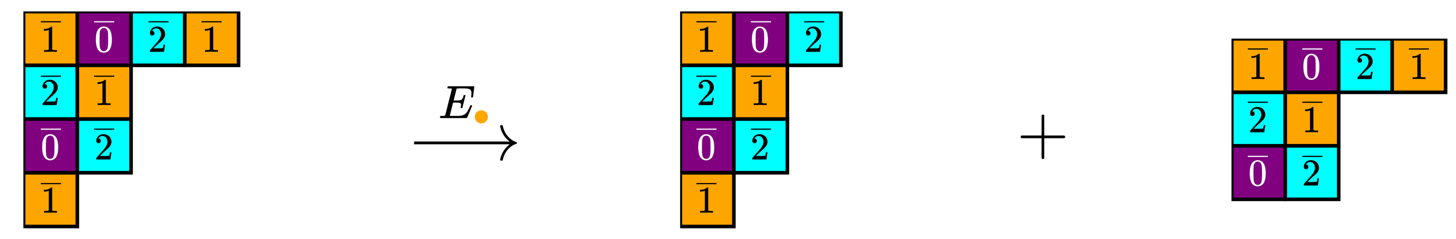


The action of $\widehat{\mathfrak{sl}_n}$ on the charged partition (c, λ) examines its *residues*:

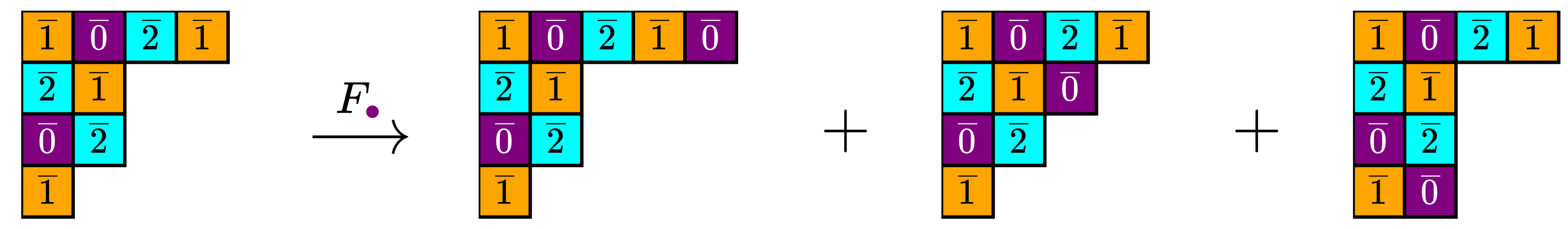
Take a charged partition $(c, \lambda) = (1, (4, 2, 2, 1))$ | Assign each cell its *content*, shifted by the charge c | Reduce modulo n to find the residues



The Chevalley generators $E_\bullet, E_\bullet, E_\bullet$ *remove* boxes of the their colour, without modifying the charge:



The Chevalley generators $F_\bullet, F_\bullet, F_\bullet$ *add* boxes their colour, without modifying the charge:



The derivation d acts on (c, λ) by counting boxes of its colour (purple), so d scales our example by 2.

The *basic representation* $V(\Lambda_0)$ of $\widehat{\mathfrak{sl}_n}$ is the submodule of $\mathcal{F}$ generated by the charge zero empty partition:

$$V(\Lambda_0) = U(\widehat{\mathfrak{sl}_n}) \cdot (0, \varnothing) \subseteq \mathcal{F}^{(0)}.$$

The shuffle relations $\mathcal{S}_n$ cut out $V(\Lambda_0)$ inside $\mathcal{F}$.

Clifford operators on Fock space

The *Clifford operators* $\psi_i, \psi_i^* \colon \mathcal{F} \to \mathcal{F}$ form the wedge or interior product with e_i.

$$\psi_i(\omega) = e_i \wedge \omega, \quad \psi_i^*(\omega) = \iota_{e_i}(\omega)$$

In terms of Maya diagrams, $\psi_i \mathsf{m}$ turns the ith bead of m black ($\psi_i \mathsf{m} = 0$ if it is already black) and multiply by a sign depending on the number of black beads to the left of i. With the m shown above, $\psi_1 \mathsf{m} = 0$ while $\psi_2 \mathsf{m}$ is the negative of the following diagram:

⋯	−6	−5	−4	−3	−2	−1	0	1	2	3	4	5	6	7	⋯
	○	○	○	●	○	○	●	●	●	●	○	●	●	●	

ψ_i^* acts similarly after swapping white with black. The Clifford operators are graded:

$$\cdots \underset{\psi_i^*}{\overset{\psi_i}{\rightleftarrows}} \mathcal{F}^{(-1)} \underset{\psi_i^*}{\overset{\psi_i}{\rightleftarrows}} \mathcal{F}^{(0)} \underset{\psi_i^*}{\overset{\psi_i}{\rightleftarrows}} \mathcal{F}^{(1)} \underset{\psi_i^*}{\overset{\psi_i}{\rightleftarrows}} \cdots$$

The shuffle equations

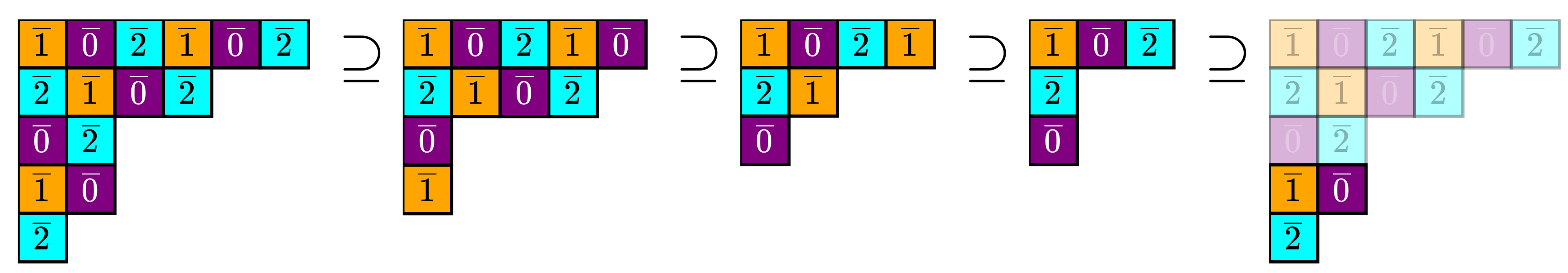
For $I \subseteq \mathbb{Z}$ and $n \in \mathbb{Z}$, set $I + n = \{i + n \mid i \in I\}$. For $d \geq 1$, define the linear map

$$\mathrm{sh}_d^n \colon \mathcal{F} \to \mathcal{F}, \quad \mathrm{sh}_d^n = \sum_{I \subseteq \mathbb{Z}, |I| = d} \psi_{I+n} \circ \psi_I^*$$

The *shuffle ideal* $\mathcal{S}_n \subseteq \Bbbk[\mathbb{P}(\mathcal{F}^{(0)})]$ cutting out the $\widehat{\mathfrak{sl}_n}$ representation $V(\Lambda_0) \subseteq \mathcal{F}^{(0)}$ is $\mathcal{S}_n = \sum_{d \geq 1} \mathrm{im}\, \mathrm{sh}_d^n$.

FLOTW multipartitions and standard monomials

By a theorem of Kostant, $\Bbbk[\mathrm{Gr}_{\mathrm{SL}_n}] \cong \bigoplus_{r \geq 0} V(r\Lambda_0)^*$, with the Cartan product as the algebra structure on the right. The work of [FLOTW99] describes a basis for $V(r\Lambda_0)$ in terms of *FLOTW multipartitions*, an r-tuple of partitions satisfying containment and *n-cylindricity*:



Above is an $(r = 4)$-multipartition $\boldsymbol{\lambda}$ satisfying containment and $(n = 3)$-cylindricity. To be *FLOTW*, the union of residues $\mathrm{Res}(\ell, \boldsymbol{\lambda})$ for each length ℓ row needs to be incomplete, for all $\ell > 1$. For $\boldsymbol{\lambda}$ above:

ℓ	6	5	4	3	2	1
$\mathrm{Res}(\ell, \boldsymbol{\lambda})$	{•}	{•}	{•, •}	{•}	{•, •, •}	{•, •, •}

and hence $\boldsymbol{\lambda}$ is not a FLOTW multipartition, as both $\mathrm{Res}(2, \boldsymbol{\lambda})$ and $\mathrm{Res}(1, \boldsymbol{\lambda})$ are complete.

References

[KLMW07] V. Kreiman, V. Lakshmibai, P. Magyar, and J. Weyman, "On ideal generators for affine Schubert varieties", *Algebraic groups and homogeneous spaces*, Tata Inst. Fund. Res. Stud. Math. **19** (2007), 353-388.

[MWY18] D. Muthiah, A. Weekes, and O. Yacobi, "The equations defining affine Grassmannians in type *A*, arXiv:1708.07076v2.

[FLOTW99] O. Foda, B. Leclerc, M. Okado, J. Thibon, and T Welsh, "Branching functions of $A_{n-1}^{(1)}$ and Jantzen-Seitz problem for Ariki-Koike algebras", .Adv. Math., 141:322–365, 1999.